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On the Solvability of Degenerated Quasilinear Elliptic Equations of Higher Order

PAVEL DRÁBEK

Department of Mathematics, University of West Bohemia, Americká 42, 30614 Plzeň, Czech Republic

ALOIS KUFNER

Mathematical Institute of Czech Academy of Sciences, Žitná 25, 11567 Praha 1, Czech Republic

AND

FRANCESCO NICOLOSI

Dipartimento di Matematica, Università di Catania Viale A. Doria, 6, 95125, Catania, Italy

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We prove the existence result for nonlinear degenerated elliptic boundary value problems of higher order. The weak solution is sought in a suitable weighted Sobolev space using the degree theory. © 1994 Academic Press, Inc.

1. INTRODUCTION

In this paper we present general existence theorem for degenerate elliptic boundary value problems for equations of the form

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, ..., D^m u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha(x) \qquad \text{in } \Omega, \quad (1.1)$$

on closed subspaces V satisfying

$$W_0^{m,p}(v, \Omega) \subseteq V \subseteq W^{m,p}(v, \Omega),$$

where $W^{m,p}(v, \Omega)$ is a *weighted Sobolev space* (see Section 2). The degeneration is determined by a vector function $v(x) = (v_\alpha(x))$, $|\alpha| = m$,

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with positive components $v_\alpha(x)$ in Ω satisfying certain integrability assumptions.

In Section 2 we define suitable weighted Lebesgue and Sobolev spaces, formulate imbedding theorems, and give sufficient conditions for the boundedness and continuity of the Němytskiĭ operator between weighted Lebesgue spaces. In Section 3 we formulate the growth assumptions on $A_\alpha(x, \xi)$ and prove that a boundary value problem for Eq. (1.1) is equivalent to a suitable operator equation on V. In Section 4 we show that under the degenerate ellipticity condition and monotonicity in the principal part it is possible to apply the degree theory for generalized monotone mappings [14, 2]. Section 5 contains existence results based on the degree theory mentioned above.

This paper may be regarded as a continuation of preceding papers [5, 3] concerning second-order degenerated quasilinear elliptic equations (where, in order to get existence results, a different approach based on the *Leray–Lions theorem* was used) and as a generalization of some results of Kufner and Opic [7, 8] (where the Dirichlet problem was considered under substantially stronger growth conditions on A_α).

2. Preliminaries

2.1. *The Němytskiĭ Operator*

We assume that $\mathbb{R}^n$ $(n \geq 1)$ is the n-dimensional Euclidean space with elements $x = (x_1, x_2, ..., x_n)$. Let Ω be an open nonempty bounded set of $\mathbb{R}^n$. Let us suppose that $\tilde{g}: \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory function and define the Němytskiĭ operator $\tilde{G}$ associated with $\tilde{g}$ by the following way

$$\tilde{G}: u(x) \to \tilde{g}(x, u_1(x), ..., u_k(x)),$$

where $u(x) = (u_1(x), ..., u_k(x))$ is a vector function. It is well known (see, e.g., [15]) that a necessary and sufficient condition for the boundedness and continuity of

$$\tilde{G}: L^{p_1}(\Omega) \times \cdots \times L^{p_k}(\Omega) \to L^{p_{k+1}}(\Omega)$$

is the following growth restriction of $\tilde{g}$

$$|\tilde{g}(x, s_1, ..., s_k)| \leq a(x) + c \sum_{i=1}^{k} |s_i|^{p_i/p_{k+1}}, \tag{2.1}$$

where $c > 0$ is a constant, $a(x) \in L^{p_{k+1}}(\Omega)$, $1 \leq p_i < +\infty$, $i = 1, ..., k+1$.

2.2. *Remark*. Let us assume that $v_0(x)$ is a positive measurable function defined in Ω. Define the *weighted Lebesgue space* $L^p(v_0, \Omega)$, $1 \leq p < +\infty$, as the space of all real-valued functions u for which

$$\|u\|_{p, v_0} = \left(\int_\Omega v_0(x) \, |u(x)|^p \, dx \right)^{1/p} < +\infty.$$

It is easily seen that the mapping $\Phi^p_{v_0}: L^p(\Omega) \to L^p(v_0, \Omega)$ defined by

$$\Phi^p_{v_0}(u) = u v_0^{-1/p}$$

is an *isometric isomorphism* between the spaces considered and connects weighted and nonweighted Lebesgue spaces (cf. Kufner and Sändig [9]).

2.3. LEMMA. *Let v_i be positive measurable functions defined in Ω, $i = 1, ..., k+1$, and let G be the Němytskiĭ operator associated with a Carathéodory function $g: \Omega \times \mathbb{R}^k \to \mathbb{R}$. Then the operator $G: L^{p_1}(v_1, \Omega) \times \cdots \times L^{p_k}(v_k, \Omega) \to L^{p_{k+1}}(v_{k+1}, \Omega)$ is bounded and continuous if and only if g satisfies*

$$|g(x, s_1, ..., s_k)| \leq a(x) \, (v_{k+1})^{-1/p_{k+1}} + c(v_{k+1})^{-1/p_{k+1}} \sum_{i=1}^{k} |s_i|^{p_i/p_{k+1}} \, (v_i)^{1/p_{k+1}}, \tag{2.2}$$

where $c > 0$ is a constant, $a(x) \in L^{p_{k+1}}(\Omega)$.

Proof. Let $\Phi^{p_i}_{v_i}$, $i = 1, 2, ..., k+1$, be the isometric isomorphisms mentioned in Remark 2.2, i.e.,

$$\Phi^{p_i}_{v_i}(u_i) = u_i v_i^{-1/p_i}.$$

Then the mapping

$$\Phi^{p_1 \cdots p_k}_{v_1 \cdots v_k}(u) = (\Phi^{p_1}_{v_1}(u_1), ..., \Phi^{p_k}_{v_k}(u_k)),$$

$u = (u_1, ..., u_k)$, is obviously an isometric isomorphism between $L^{p_1}(\Omega) \times \cdots \times L^{p_k}(\Omega)$ and $L^{p_1}(v_1, \Omega) \times \cdots \times L^{p_k}(v_k, \Omega)$. Using the notation from Section 2.1, we have

$$\begin{array}{ccc}
L^{p_1}(\Omega) \times \cdots \times L^{p_k}(\Omega) & \xrightarrow{\quad \tilde{G} \quad} & L^{p_{k+1}}(\Omega) \\
\Big\uparrow {\scriptstyle (\Phi^{p_1 \cdots p_k}_{v_1 \cdots v_k})^{-1}} & & \Big\downarrow {\scriptstyle \Phi^{p_{k+1}}_{v_{k+1}}} \\
L^{p_1}(v_1, \Omega) \times \cdots \times L^{p_k}(v_k, \Omega) & \xrightarrow{\quad G \quad} & L^{p_{k+1}}(v_{k+1}, \Omega)
\end{array}$$

We have

$$G = \Phi^{p_{k+1}}_{v_{k+1}} \circ \tilde{G} \circ (\Phi^{p_1 \cdots p_k}_{v_1 \cdots v_k})^{-1},$$

i.e.,

$$g(x, s_1, \ldots, s_k) = (v_{k+1})^{-1/p_{k+1}} \tilde{g}(x, s_1 v_1^{1/p_1}, \ldots, s_k v_k^{1/p_k}). \tag{2.3}$$

Hence G is bounded and continuous if and only if $\tilde{G}$ is bounded and continuous and the assertion follows from (2.3) and (2.1).

2.4. Lemma *The operator* $G\colon L^{p_1}(v_1, \Omega) \times \cdots \times L^{p_k}(v_k, \Omega) \to [L^p(v_0, \Omega)]^*$ *is bounded and continuous if and only if* g *satisfies*

$$|g(x, s_1, \ldots, s_k)| \leqslant a(x)(v_0)^{1/p} + c(v_0)^{1/p} \sum_{i=1}^{k} |s_i|^{p_i(p-1)/p} (v_i)^{(p-1)/p}, \tag{2.4}$$

where $c > 0$ *is a constant,* $a(x) \in L^{p/(p-1)}(\Omega)$.

Proof. For w measurable and positive in Ω, it is

$$[L^p(w, \Omega)]^* = L^{p'}(w^*, \Omega) \tag{2.5}$$

with $p' = p/(p-1)$, $w^* = w^{-1/(p-1)}$. Indeed: for $u \in L^p(w, \Omega)$ and $\varphi \in L^{p'}(w^*, \Omega)$, the Hölder inequality yields

$$\begin{aligned}|\langle F, u \rangle| &= \left| \int_\Omega u\varphi \, dx \right| = \left| \int_\Omega w^{1/p} u w^{-1/p} \varphi \, dx \right| \\ &\leqslant \left(\int_\Omega w\, |u|^p \, dx \right)^{1/p} \left(\int_\Omega w^{-p'/p} |\varphi|^{p'} \, dx \right)^{1/p'} = \|u\|_{p,w} \cdot \|\varphi\|_{p',w^*} < \infty.\end{aligned}$$

On the other hand, $u \in L^p(w, \Omega)$ if and only if $u \in L^p(\Omega; d\mu)$ with the measure μ given by

$$\mu(E) = \int_E w \, dx, \qquad E \subseteq \Omega,$$

and for every $F \in [L^p(\Omega; d\mu)]^* = [L^p(w, \Omega)]^*$, there is a function $V \in L^{p'}(\Omega; d\mu)$ such that

$$\langle F, u \rangle = \int_\Omega uV \, d\mu = \int_\Omega uVw \, dx = \int_\Omega uv \, dx$$

with $v = Vw$. But $v \in L^{p'}(w^*, \Omega)$ since

$$\|v\|_{p',w^*}^{p'} = \int_\Omega |Vw|^{p'} w^{-1/(p-1)} \, dx = \int_\Omega |V|^{p'} w \, dx = \int_\Omega |V|^{p'} \, d\mu < \infty.$$

Thus, we have (2.5). The proof of our assertion now follows immediately from Lemma 2.3 if we set $p_{k+1} = p$, $v_{k+1} = (v_0)^{-1/(p-1)}$ in (2.2).

2.5. *Weighted Sobolev Spaces*

Denote by $M(j)$ the number of distinct multiindices $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integer components α_i and with the length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ not exceeding j. For a given differentiable function u defined in Ω we denote

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad D^k u = \{D^\alpha u;\ |\alpha| = k\}.$$

Let us suppose that $p > 1$ is a real number, $m \geqslant 1$ is an integer, and $v(x) = (v_\alpha(x))$, $|\alpha| = m$ is a vector function with $M(m) - M(m-1)$ components. Further we suppose that every component $v_\alpha(x)$ is a measurable function which is positive a.e. in Ω and satisfies

$$v_\alpha(x) \in L^1_{\mathrm{loc}}(\Omega), \tag{2.6}$$

$$\frac{1}{v_\alpha(x)} \in L^{1/(p-1)}_{\mathrm{loc}}(\Omega) \tag{2.7}$$

for any $|\alpha| = m$.

Now, we denote by $W^{m,p}(v, \Omega)$ the space of all real-valued functions u such that the derivatives in the sense of distributions fulfill

$$D^\beta u \in L^p(\Omega) \qquad \text{for all} \quad |\beta| \leqslant m - 1$$

and

$$(v_\alpha)^{1/p} D^\alpha u \in L^p(\Omega) \qquad \text{for all} \quad |\alpha| = m.$$

The condition (2.6) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{m,p}(v, \Omega)$ and consequently, we can introduce the subspace $W_0^{m,p}(v, \Omega)$ of $W^{m,p}(v, \Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{m,p,v} = \left[\sum_{|\beta| \leqslant m-1} \int_\Omega |D^\beta u(x)|^p\, dx + \sum_{|\alpha| = m} \int_\Omega v_\alpha(x)\, |D^\alpha u(x)|^p\, dx \right]^{1/p}.$$

Moreover, condition (2.7) implies that $W^{m,p}(v, \Omega)$ as well as $W_0^{m,p}(v, \Omega)$ are (*reflexive*) *Banach spaces* (cf. Kufner and Sändig [9]).

2.6. LEMMA. *Let us suppose*

$$\frac{1}{v_\alpha(x)} \in L^{g^*}(\Omega) \tag{2.8}$$

for any $|\alpha| = m$, *with some* $g^* \geqslant 1/(p-1)$. *Then the space* $W^{m,p}(v, \Omega)$ *is continuously imbedded into* $W^{m,p_1}(\Omega)$, *where*

$$p_1 = \frac{pg^*}{g^*+1}. \tag{2.9}$$

2.7. LEMMA. *Let us suppose* (2.8) *and denote*

$$\kappa_1 = \frac{mpg^* - n(g^*+1)}{pg^*} = m - \frac{n}{p_1} \tag{2.10}$$

with p_1 *given by* (2.9). *Then the following continuous imbeddings hold with* $k < m$ *a nonnegative integer, provided* Ω *has the so called cone property*:

(i) *for* $k < \kappa_1$,

$$W^{m,p}(v, \Omega) \hookrightarrow C^k(\bar{\Omega});$$

(ii) *for* $k = \kappa_1$,

$$W^{m,p}(v, \Omega) \hookrightarrow W^{k,r}(\Omega)$$

with arbitrary r, $1 < r < +\infty$;

(iii) *for* $k > \kappa_1$,

$$W^{m,p}(v, \Omega) \hookrightarrow W^{k,r_k}(\Omega),$$

where r_k *satisfies*

$$1 < r_k \leqslant q_k = \frac{pg^*n}{n(g^*+1) - pg^*(m-k)}.$$

Moreover, the imbeddings (i) *and* (ii) *are compact and the imbedding* (iii) *is compact if* $r_k < q_k$.

The proof of Lemma 2.7 follows from Lemma 2.6 and Sobolev imbedding theorems (see, e.g., [1, 6, 13]). For the precise proofs of Lemmas 2.6 and 2.7 see [10].

2.8. *Remark.* If we suppose that the number g^* from Lemma 2.6 satisfies

$$g^* > \frac{n}{p} \tag{2.11}$$

then $q_k > p$ for any $k < m$. In particular, due to Lemma 2.7(iii), the space $W^{m,p}(v, \Omega)$ is then *compactly imbedded* into $W^{m-1,p}(\Omega)$.

In the sequel, we suppose that (2.11) holds. In order to fulfill also assumption (2.7), we suppose that

$$g^* > \max\left\{\frac{n}{p}, \frac{1}{p-1}\right\}, \tag{2.12}$$

which guarantees that both conditions (2.7) and (2.11) are satisfied.

3. Operator Representation

3.1. *The Right-Hand Side of* (1.1)

Let f_α be the functions appearing in the right-hand side of (1.1). Suppose that g^* satisfies (2.11) and let κ_1 be given by (2.10). We assume that the functions f_α, $|\alpha| \leqslant m$, satisfy the following *conditions*:

(a) $f_\alpha(x) \in L^{p/(p-1)}(v_\alpha^{-1/(p-1)}, \Omega)$ *if* $|\alpha| = m$;

(b) $f_\alpha(x) \in L^{Z_\alpha}(\Omega)$ if $\kappa_1 < |\alpha| \leqslant m-1$, where

$$Z_\alpha = \frac{pg^*n}{pg^*n - n(g^*+1) + pg^*(m-|\alpha|)};$$

(c) $f_\alpha(x) \in L^Z(\Omega)$ if $|\alpha| = \kappa_1$ where $1 < Z < +\infty$ is arbitrary;

(d) $f_\alpha \in L^1(\Omega)$ if $|\alpha| < \kappa_1$.

3.2. *The Left-Hand Side of* (1.1)

Assume that $A_\alpha(x, \xi)$, $|\alpha| \leqslant m$, are Carathéodory functions with $x \in \Omega$ and $\xi \in \mathbb{R}^{M(m)}$. Let $g_1 : \mathbb{R} \to \mathbb{R}$ be a positive, continuous, and nondecreasing function, $a_\alpha(x) \in L^{p/(p-1)}(\Omega)$, $|\alpha| = m$, $a_\alpha(x) \in L^{p/q_\alpha}(\Omega)$, $\kappa_1 < |\alpha| < m$, $a_\alpha(x) \in L^{1/\sigma}(\Omega)$, $|\alpha| = \kappa_1$, and $a_\alpha(x) \in L^1(\Omega)$, $|\alpha| < \kappa_1$ (see below). We suppose that $A_\alpha(x, \xi)$, $|\alpha| \leqslant m$, satisfy the following *growth conditions*:

(i) Let $|\alpha| = m$. Then we assume

$$|A_\alpha(x, \xi)| \leqslant g_1(|\xi_0|)\, v_\alpha^{1/p} \times \left[a_\alpha(x) + \sum_{\kappa_1 \leqslant |\beta| < m} |\xi_\beta|^{r_\beta} + \sum_{|\beta| = m} (v_\beta)^{(p-1)/p} |\xi_\beta|^{p-1} \right],$$

where $\xi_0 = \{\xi_\beta;\, |\beta| < \kappa_1\}$,

$$1 < r_\beta < \frac{(p-1)g^*n}{n(g^*+1) - pg^*(m-|\beta|)} \qquad \text{if} \quad \kappa_1 < |\beta| < m,$$

$1 < r_\beta < +\infty$ is arbitrary if $|\beta| = \kappa_1$.

(ii) Let $\kappa_1 < |\alpha| < m$. Then we assume

$$|A_\alpha(x, \xi)| \leqslant g_1(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_1 \leqslant |\beta| < m} |\xi_\beta|^{s_{\alpha\beta}} + \sum_{|\beta| = m} v_\beta^{q_\alpha/p} |\xi_\beta|^{q_\alpha}\right],$$

where

$$1 < q_\alpha < \frac{pg^*n - n(g^*+1) + pg^*(m - |\alpha|)}{g^*n},$$

$$1 < s_{\alpha\beta} = \frac{r_{|\beta|}}{r'_{|\alpha|}} < \frac{pg^*n - n(g^*+1) + pg^*(m - |\alpha|)}{n(g^*+1) - pg^*(m - |\beta|)} \qquad \text{if} \quad \kappa_1 < |\beta| < m,$$

$1 < s_{\alpha\beta} < +\infty$ is arbitrary if $|\beta| = \kappa_1$.

(iii) Let $|\alpha| = \kappa_1$. Then we assume

$$|A_\alpha(x, \xi)| \leqslant g_1(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_1 \leqslant |\beta| < m} |\xi_\beta|^{\tilde{s}_\beta} + \sum_{|\beta| = m} v_\beta^{\sigma} |\xi_\beta|^{p\sigma}\right],$$

where $\tilde{s}_\beta$, σ are real numbers which satisfy the inequalities

$$0 < \sigma < 1,$$

$$1 < \tilde{s}_\beta = \frac{p}{p - q_\beta} < \frac{pg^*n}{n(g^*+1) - pg^*(m - |\beta|)} \qquad \text{if} \quad \kappa_1 < |\beta| < m,$$

$1 < s_\beta < +\infty$ is arbitrary if $|\beta| = \kappa_1$.

(iv) Let $|\alpha| < \kappa_1$. Then we assume

$$|A_\alpha(x, \xi)| \leqslant g_1(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_1 \leqslant |\beta| < m} |\xi_\beta|^{\tilde{s}_\beta} + \sum_{|\beta| = m} v_\beta |\xi_\beta|^{p}\right].$$

Note that $\xi_0 = \varnothing$ for $\kappa_1 \leqslant 0$. In this case we take $g_1(t) = 1$.

3.3. *Remark.* Let us note that if $g^* \to +\infty$ in the conditions (i)–(iv) and (a)–(d), we obtain the *standard growth assumptions* for *nondegenerate* equations of order $2m$ in divergent form (cf. [14, 4]).

3.4. *An Operator Associated with* (1.1)

Let V be a closed subspace of $W^{m,p}(v, \Omega)$ such that

$$W_0^{m,p}(v, \Omega) \subseteq V \subseteq W^{m,p}(v, \Omega).$$

When dealing with $V \neq W_0^{m,p}(v, \Omega)$ we always assume that Ω satisfies the *cone property* (see, e.g., [1]). Let us define the operator

$$T \colon V \to V^*$$

by

$$\langle Tu, \varphi \rangle = \sum_{|\alpha| \leqslant m} \int_\Omega [A_\alpha(x, u, ..., D^m u) - f_\alpha(x)] \, D^\alpha \varphi(x) \, dx \tag{3.1}$$

for any $\varphi \in V$, where $\langle \cdot, \cdot \rangle$ denotes the duality between V^* and V.

In fact the subspace V is determined by the type of *homogeneous boundary conditions* appearing in the boundary value problem for Eq. (1.1). The case $V = W_0^{m,p}(v, \Omega)$ corresponds to the *Dirichlet problem* (where formally $D^\beta u = 0$ on $\partial\Omega$ for $|\beta| \leqslant m-1$) and $V = W^{m,p}(v, \Omega)$ corresponds to the *Neumann problem* (where formally $D^\beta u = 0$ on $\partial\Omega$ for $m \leqslant |\beta| \leqslant 2m-1$). We set

$$\langle Tu, \varphi \rangle = \sum_{|\alpha| \leqslant m} \int_\Omega [A_\alpha(x, u+u_0, ..., D^m(u+u_0)) - f_\alpha(x)] \, D^\alpha \varphi(x) \, dx$$

when dealing with the *nonhomogeneous boundary* conditions.

3.5. THEOREM. *Let us assume that conditions* (a)–(d) *from Section* 3.1 *and* (i)–(iv) *from Section* 3.2 *are satisfied. Then the operator T defined in* (3.1) *is bounded and continuous.*

Proof. Since $u \in W^{m,p}(v, \Omega)$ means that $D^\beta u \in L^p(v_\beta, \Omega)$ for $|\beta| \leqslant m$ with $v_\beta(x) = 1$ if $|\beta| < m$, we can identify the space $W^{m,p}(v, \Omega)$ with the product space

$$P_1 = \prod_{|\beta| \leqslant m} L^p(v_\beta, \Omega).$$

More precisely, assuming that (2.12) holds, we can use Lemma 2.7 and identify $W^{m,p}(v, \Omega)$ with the product space

$$P = \prod_{|\beta| \leqslant m} X_\beta,$$

where $X_\beta = C(\bar{\Omega})$ for $|\beta| < \kappa_1$, $X_\beta = L^r(\Omega)$ with r arbitrary, $1 \leqslant r < +\infty$, for $|\beta| = \kappa_1$, $X_\beta = L^{\tilde{s}_\beta}(\Omega)$ with $\tilde{s}_\beta$ such that

$$1 < \tilde{s}_\beta < \frac{pg^* n}{n(g^*+1) - pg^*(m - |\beta|)} \qquad \text{for} \quad \kappa_1 < |\beta| < m,$$

and $X_\beta = L^p(v_\beta, \Omega)$ for $|\beta| = m$. Using now Lemma 2.4, we can show that the Němytskiĭ operator G_α associated with the "coefficient" $A_\alpha(x, \xi)$ appearing in (1.1) is bounded and continuous

(I) from P into $[L^p(v_\alpha, \Omega)]^*$ if $|\alpha| = m$,

(II) from P into $[L^{\tilde{s}_\alpha}(\Omega)]^*$,

(III) from P into $[L^g(\Omega)]^*$ with arbitrary g, $1 < g < \infty$ if $|\alpha| = \kappa_1$,

(IV) from P into $L^1(\Omega) \subset [L^\infty(\Omega)]^*$ if $|\alpha| < \kappa_1$.

Splitting the summation in (3.1) so that

$$\sum_{|\alpha| \leq m} = \sum_{|\alpha| = m} + \sum_{\kappa_1 < |\alpha| < m} + \sum_{|\alpha| = \kappa_1} + \sum_{|\alpha| < \kappa_1},$$

we obtain the assertion using additionally conditions (a)–(d) from Section 3.1.

4. Degree Theory of Generalized Monotone Mappings

4.1. *Ellipticity Condition*

Let $g_2 : \mathbb{R} \to \mathbb{R}$ be a continuous, positive, and nonincreasing function and $g_3 : \mathbb{R} \to \mathbb{R}$ be a continuous, positive, and nondecreasing function. Let us denote $M_0 = M(m) - M(m-1)$. We suppose that the following *ellipticity condition* is satisfied

$$\sum_{|\alpha| = m} A_\alpha(x, \eta, \zeta)\, \zeta_\alpha \geq g_2(|\eta_0|) \sum_{|\beta| = m} v_\beta\, |\zeta_\beta|^p - g_3(|\eta_0|) \sum_{\kappa_1 \leq |\beta| < m} |\eta_\beta|^{\tilde{s}_\beta} \tag{4.1}$$

for a.e. $x \in \Omega$, for any $\zeta = \{\zeta_\alpha; |\alpha| = m\}$, $\eta = \{\eta_\beta; |\beta| < m\}$, $\zeta \in \mathbb{R}^{M_0}$, $\eta \in \mathbb{R}^{M(m-1)}$, where $\eta_0 = \{\eta_\beta; |\beta| < \kappa_1\}$ and $\tilde{s}_\beta$ is defined in 3.2(iii).

4.2. *Monotonicity Condition*

We assume that the principal part of the equation satisfies

$$\sum_{|\alpha| = m} [A_\alpha(x, \eta, \zeta) - A_\alpha(x, \eta, \zeta')](\zeta_\alpha - \zeta'_\alpha) > 0 \tag{4.2}$$

for a.e. $x \in \Omega$, for any $\eta = \{\eta_\beta; |\beta| \leq m-1\}$ and for any $\zeta, \zeta' \in \mathbb{R}^{M_0}$ such that $\zeta \neq \zeta'$.

4.3. *Degree Theory*

In order to obtain the existence results for Eq. (1.1) we apply the *degree theory of generalized monotone mappings* presented in the book of Skrypnik

[14]. To this purpose we must verify that the operator T defined in (3.1) satisfies the condition

$\alpha(V)$: Let $\{u_n\}$ be an arbitrary sequence in V such that $\{u_n\}$ converges weakly to u_0 in V and

$$\limsup_{n\to\infty} \langle Tu_n, u_n - u_0\rangle \leqslant 0. \tag{4.3}$$

Then $\{u_n\}$ converges strongly to u_0 in V.

4.4. LEMMA. *Assume that the assumptions* (a)–(d) *from Section* 3.1, (i)–(iv) *from Section* 3.2, (4.1), *and* (4.2) *are satisfied. Then the operator* $T: V \to V^*$ *defined in* (3.1) *satisfies condition* $\alpha(V)$.

Proof. Let us suppose that u_n converge to u_0 weakly in V and (4.3) holds. It follows from the weak convergence $u_n \to u_0$ in V the strong convergence $D^\beta u_n \to D^\beta u_0$ in $L^{\bar{s}_\beta}(\Omega)$ for $\kappa_1 \leqslant |\beta| \leqslant m-1$ and in $C(\bar{\Omega})$ for $|\beta| < \kappa_1$ (see Lemma 2.7). Assumptions (ii)–(iv) from Section 3.2 then imply

$$\sum_{|\alpha|\leqslant m-1} \int_\Omega A_\alpha(x, u_n, ..., D^m u_n)\, D^\alpha(u_n - u_0)\, dx \to 0$$

for $n \to \infty$. Let us write

$$\begin{aligned}
&\sum_{|\alpha|=m} \int_\Omega A_\alpha(x, u_n, ..., D^{m-1}u_n, D^m u_0)\, D^\alpha(u_n - u_0)\, dx \\
&\quad= \sum_{|\alpha|=m} \int_\Omega [A_\alpha(x, u_n, ..., D^{m-1}u_n, D^m u_0) \\
&\qquad - A_\alpha(x, u_0, ..., D^{m-1}u_0, D^m u_0)]\, D^\alpha(u_n - u_0)\, dx \\
&\qquad + \sum_{|\alpha|=m} \int_\Omega A_\alpha(x, u_0, ..., D^{m-1}u_0, D^m u_0)\, D^\alpha(u_n - u_0)\, dx.
\end{aligned} \tag{4.4}$$

Growth assumptions (ii)–(iv) from Section 3.2, the strong convergence $D^\beta u_n \to D^\beta u_0$ in $L^{\bar{s}_\beta}(\Omega)$ for $\kappa_1 \leqslant |\beta| \leqslant m-1$ and in $C(\bar{\Omega})$ for $|\beta| < \kappa_1$, and the continuity of the Němytskiĭ operator G_α from $\prod_{|\beta|<m} X_\beta$ into $[L^p(v_\alpha, \Omega)]^*$ (see the proof of Theorem 3.5) imply

$$\begin{aligned}
&\sum_{|\alpha|=m} \int_\Omega [A_\alpha(x, u_n, ..., D^{m-1}u_n, D^m u_0) \\
&\qquad - A_\alpha(x, u_0, ..., D^{m-1}u_0, D^m u_0)]\, D^\alpha(u_n - u_0)\, dx \to 0.
\end{aligned} \tag{4.5}$$

By virtue of the weak convergence of $\{u_n\}$ to u_0 we have

$$\sum_{|\alpha|=m} \int_\Omega A_\alpha(x, u_0, ..., D^{m-1}u_0, D^m u_0)\, D^\alpha(u_n - u_0)\, dx \to 0. \tag{4.6}$$

It follows from (4.4)–(4.6) that

$$\sum_{|\alpha|=m} \int_\Omega A_\alpha(x, u_n, ..., D^{m-1}u_n, D^m u_0)\, D^\alpha(u_n - u_0)\, dx \to 0. \tag{4.7}$$

Hence it follows from (4.3) and (4.7) that

$$\limsup_{n\to\infty} \int_\Omega \sum_{|\alpha|=m} [A_\alpha(x, u_n, ..., D^m u_n) - A_\alpha(x, u_n, ..., D^{m-1}u_n, D^m u_0)]\, D^\alpha(u_n - u_0)\, dx \leqslant 0. \tag{4.8}$$

We show that it follows from (4.8) that

(A) the equality

$$\lim_{\text{meas}\, E \to 0} \int_E \sum_{|\alpha|=m} v_\alpha(x)\, |D^\alpha u_n(x)|^p\, dx = 0, \qquad E \subseteq \Omega,$$

holds uniformly with respect to n;

(B) for $|\alpha| = m$ the sequence $\{v_\alpha^{1/p} D^\alpha u_n\}$ converges to $v_\alpha^{1/p} D^\alpha u_0$ in measure.

Suppose for a moment that (A) and (B) hold. Then it follows that $\{v_\alpha^{1/p} D^\alpha u_n\}$ converges to $v_\alpha^{1/p} D^\alpha u_0$ strongly in $L^p(\Omega)$; i.e., $\{u_n\}$ converges strongly to u_0 in V which completes the proof of Lemma 4.4.

Let us prove (A). Let us denote

$$\lambda_n(E) = \int_E \sum_{|\alpha|=m} \{A_\alpha(x, u_n, ..., D^m u_n) - A_\alpha(x, u_n, ..., D^{m-1}u_n, D^m u_0)\}\, D^\alpha(u_n - u_0)\, dx$$

for arbitrary measurable set $E \subseteq \Omega$. It follows from (4.2) that

$$\lambda_n(E) \to 0 \quad \text{for} \quad n \to \infty \quad \text{and} \quad \lambda_n(E) \leqslant \lambda_n(\Omega) \tag{4.9}$$

for any measurable set $E \subseteq \Omega$. The following estimate holds due to the ellipticity condition (4.1) and growth assumption (i) from Section 3.2:

$$\lambda_n(E) \geqslant \int_E \sum_{|\alpha|=m} A_\alpha(x, u_n, ..., D^m u_n)\, D^\alpha u_n\, dx$$

$$- \int_E \sum_{|\alpha|=m} |A_\alpha(x, u_n, ..., D^{m-1} u_n, D^m u_0)|\, (|D^\alpha u_n| + |D^\alpha u_0|)\, dx$$

$$- \int_E \sum_{|\alpha|=m} |A_\alpha(x, u_n, ..., D^m u_n)|\, |D^\alpha u_0|\, dx$$

$$\geqslant c_2 \int_E \sum_{|\alpha|=m} v_\alpha(x)\, |D^\alpha u_n|^p\, dx - c_3 \int_E \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} |D^\beta u_n|^{\bar{s}_\beta}\, dx$$

$$- c_1 \sum_{|\alpha|=m} \int_E \Bigg[v_\alpha^{1/p} a_\alpha + \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} v_\alpha^{1/p}\, |D^\beta u_n|^{r_\beta}$$

$$+ \sum_{|\beta|=m} v_\alpha^{1/p} v_\beta^{(p-1)/p}\, |D^\beta u_0|^{p-1} \Bigg] (|D^\alpha u_n| + |D^\alpha u_0|)\, dx$$

$$- c_1 \sum_{|\alpha|=m} \int_E \Bigg[v_\alpha^{1/p} a_\alpha + \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} v_\alpha^{1/p}\, |D^\beta u_n|^{r_\beta}$$

$$+ \sum_{|\beta|=m} v_\alpha^{1/p} v_\beta^{(p-1)/p}\, |D^\beta u_n|^{p-1} \Bigg] |D^\alpha u_0|\, dx. \tag{4.10}$$

Note that the existence of constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ follows from the compact imbedding of lower-order terms (with $|\beta| < \kappa_1$) into $C(\bar{\Omega})$ and from the properties of functions g_1, g_2, g_3. It follows from (4.10) that there are constants $c_4 > 0$, $c_5 > 0$ such that

$$\int_E \sum_{|\alpha|=m} v_\alpha(x)\, |D^\alpha u_n(x)|^p\, dx$$

$$\leqslant c_4 \lambda_n(E) + c_5 \int_E \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} |D^\beta u_n|^{\bar{s}_\beta}\, dx$$

$$+ c_5 \sum_{|\alpha|=m} \int_E \Bigg[v_\alpha^{1/p} a_\alpha + \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} v_\alpha^{1/p}\, |D^\beta u_n|^{r_\beta}$$

$$+ \sum_{|\beta|=m} v_\alpha^{1/p} v_\beta^{(p-1)/p}\, |D^\beta u_0|^{p-1} \Bigg] (|D^\alpha u_n| + |D^\alpha u_0|)\, dx$$

$$+ c_5 \sum_{|\alpha|=m} \int_E \Bigg[v_\alpha^{1/p} a_\alpha + \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} v_\alpha^{1/p}\, |D^\beta u_n|^{r_\beta}$$

$$+ \sum_{|\beta|=m} v_\alpha^{1/p} v_\beta^{(p-1)/p}\, |D^\beta u_n|^{p-1} \Bigg] |D^\alpha u_0|\, dx. \tag{4.11}$$

We use the *Young inequality*

$$ab \leqslant \frac{\varepsilon^\tau a^\tau}{\tau} + \frac{b^{\tau'}}{\varepsilon^{\tau'}\tau'}, \tag{4.12}$$

where a, b, ε, τ are positive reals, $1/\tau + 1/\tau' = 1$.

Let $|\beta| \leqslant m-1$. Then applying (4.12) with $\tau = p$, $a = |D^\alpha u_n| \, v_\alpha^{1/p}$, $b = |D^\beta u_n|^{r_\beta}$, we get

$$\begin{aligned}\int_E |D^\beta u_n|^{r_\beta} \, |D^\alpha u_n| \, v_\alpha^{1/p} \, dx \\ \leqslant \frac{\varepsilon^p}{p} \int_E |D^\alpha u_n|^p \, v_\alpha \, dx + \frac{1}{\varepsilon^{p'} p'} \int_E |D^\beta u_n|^{\tilde{s}_\beta} \, dx,\end{aligned} \tag{4.13}$$

because $p' r_\beta = p r_\beta/(p-1) = \tilde{s}_\beta$.

Similarly we get

$$\begin{aligned}\int_E |D^\beta u_n|^{r_\beta} \, |D^\alpha u_0| \, v_\alpha^{1/p} \, dx \\ \leqslant \frac{\varepsilon^p}{p} \int_E |D^\alpha u_0|^p v_\alpha \, dx + \frac{1}{\varepsilon^{p'} p'} \int_E |D^\beta u_n|^{\tilde{s}_\beta} \, dx.\end{aligned} \tag{4.14}$$

Let $|\beta| = m$. Then applying (4.12) with $\tau = p$, $a = |D^\alpha u_n| \, v_\alpha^{1/p}$, $b = v_\beta^{(p-1)/p} \, |D^\beta u_0|^{p-1}$ we get

$$\begin{aligned}\int_E v_\beta^{(p-1)/p} \, |D^\beta u_0|^{p-1} \, v_\alpha^{1/p} \, |D^\alpha u_n| \, dx \\ \leqslant \frac{\varepsilon^p}{p} \int_E |D^\alpha u_n|^p \, v_\alpha \, dx + \frac{1}{\varepsilon^{p'} p'} \int_E |D^\beta u_0|^p \, v_\beta \, dx.\end{aligned} \tag{4.15}$$

Similarly we get the following estimates:

$$\begin{aligned}\int_E v_\beta^{(p-1)/p} \, |D^\beta u_0|^{p-1} \, v_\alpha^{1/p} \, |D^\alpha u_0| \, dx \\ \leqslant \frac{\varepsilon^p}{p} \int_E |D^\alpha u_0|^p \, v_\alpha \, dx + \frac{1}{\varepsilon^{p'} p'} \int_E |D^\beta u_0|^p \, v_\beta \, dx\end{aligned} \tag{4.16}$$

and $\left(\text{with } \varepsilon \text{ replaced by } \dfrac{1}{\varepsilon}\right)$

$$\begin{aligned}\int_E v_\beta^{(p-1)/p} \, |D^\beta u_n|^{p-1} \, v_\alpha^{1/p} \, |D^\alpha u_0| \, dx \\ \leqslant \frac{\varepsilon^{p'}}{p'} \int_E |D^\beta u_n|^p \, v_\beta \, dx + \frac{1}{\varepsilon^p \cdot p} \int_E |D^\alpha u_0|^p \, v_\alpha \, dx.\end{aligned} \tag{4.17}$$

Finally, we apply (4.12) with $\tau = p$, $a = v_\alpha^{1/p} |D^\alpha u_n|$, $b = 1$ and we get

$$\int_E v_\alpha^{1/p} |D^\alpha u_n| \, dx \leqslant \frac{\varepsilon^p}{p} \int_E |D^\alpha u_n|^p \, v_\alpha \, dx + \frac{\|a_\alpha\|_{p'}^{p'}}{\varepsilon^{p'} p'} \tag{4.18}$$

and similarly

$$\int_E v_\alpha^{1/p} |D^\alpha u_0| \, dx \leqslant \frac{\varepsilon^p}{p} \int_E |D^\alpha u_0|^p \, v_\alpha \, dx + \frac{\|a_\alpha\|_{p'}^{p'}}{\varepsilon^{p'} p'}. \tag{4.19}$$

It follows from (4.11) and (4.13)–(4.19) that taking $\varepsilon > 0$ small enough there are positive constants c_6, c_7, c_8 such that

$$\begin{aligned} \int_E \sum_{|\alpha| = m} v_\alpha |D^\alpha u_n|^p \, dx &\leqslant c_6 \lambda_n(E) \\ &+ c_7 \int_E \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} |D^\beta u_n|^{s_\beta} \, dx + c_8 \int_E \sum_{|\beta| = m} v_\beta |D^\beta u_0|^p \, dx. \end{aligned} \tag{4.20}$$

Assume that meas $E \to 0$. Then

$$\int_E \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} |D^\beta u_n|^{s_\beta} \, dx \to 0 \tag{4.21}$$

by virtue of the strong convergence of $D^\beta u_n$ to $D^\beta u_0$ in $L^{s_\beta}(\Omega)$. Since $u_0 \in V$ is fixed, we also have

$$\int_E \sum_{|\beta| = m} v_\beta |D^\beta u_0|^p \, dx \to 0. \tag{4.22}$$

Now, (4.9) and (4.20)–(4.22) imply

$$\int_E \sum_{|\alpha| = m} v_\alpha |D^\alpha u_n|^p \, dx \to 0,$$

which proves (A).

Let us prove (B). Let ε, δ be arbitrary positive numbers. Set

$$F_{\varepsilon, n} = \left\{ x \in \Omega ; \sum_{|\alpha| = m} (v_\alpha)^{1/p} |D^\alpha u_n(x) - D^\alpha u_0(x)| \geqslant \varepsilon \right\}.$$

We choose open sets $E_\delta^{(1)}$, $E_\delta^{(2)}$ in such a way that meas $E_\delta^{(i)} < \delta/4$, $i = 1, 2$, and

$$K_\delta = \sup \left\{ \sum_{|\alpha| = m} |D^\alpha u_0(x)| + \sum_{|\alpha| \leqslant m-1} |D^\alpha u_n(x)|; \right.$$

$$\left. x \in \Omega \backslash E_\delta^{(1)} \right\} < +\infty, \tag{4.23}$$

$$K_\varepsilon^\delta = \inf \left\{ \sum_{|\alpha| = m} [A_\alpha(x, \eta, \zeta) - A_\alpha(x, \eta, \zeta')](\zeta_\alpha - \zeta'_\alpha); \right.$$

$$\left. x \in \Omega \backslash E_\delta^{(2)},\ |\eta| \leqslant K_\delta,\ |\zeta'| \leqslant K_\delta,\ \sum_{|\alpha| = m} v_\alpha^{1/p}(x) |\zeta_\alpha - \zeta'_\alpha| \geqslant \varepsilon \right\} > 0. \tag{4.24}$$

Note that the existence of $E_\delta^{(1)}$ satisfying (4.23) follows from Luzin's theorem (see, e.g., [12, 6]), the existence of $E_\delta^{(2)}$ satisfying (4.24) follows from Luzin's theorem and from the monotonicity condition (4.2). It follows from (4.9) that

$$\begin{aligned}
\lambda_n(\Omega) &\geqslant \lambda_n(F_{\varepsilon, n} \backslash (E_\delta^{(1)} \cup E_\delta^{(2)})) \\
&= \int_{F_{\varepsilon, n} \backslash (E_\delta^{(1)} \cup E_\delta^{(2)})} \sum_{|\alpha| = m} [A_\alpha(x, u_n, \ldots, D^m u_n) \\
&\quad - A_\alpha(x, u_n, \ldots, D^{m-1} u_n, D^m u_0)] \cdot D^\alpha(u_n - u_0)\, dx \\
&\geqslant K_\varepsilon^\delta \left(\operatorname{meas} F_{\varepsilon, n} - \frac{\delta}{2} \right).
\end{aligned}$$

Hence

$$\operatorname{meas} F_{\varepsilon, n} \leqslant \frac{\delta}{2} + \frac{\lambda_n(\Omega)}{K_\varepsilon^\delta}. \tag{4.25}$$

Due to (4.9) we can find n so large that

$$\frac{\lambda_n(\Omega)}{K_\varepsilon^\delta} < \frac{\delta}{2},$$

which together with (4.25) yields meas $F_{\varepsilon, n} < \delta$. Hence $\{v_\alpha^{1/p} D^\alpha u_n\}$ converges to $v_\alpha^{1/p} D^\alpha u_0$ in measure; i.e., condition (B) holds.

4.5. *Remark.* It follows from Theorem 3.5 and Lemma 4.4 that the degree of the mapping $T: V \to V^*$ may be defined in the same way as in [14]. Note that the properties of this degree are similar to those of the Leray–Schauder degree.

5. Existence Results

5.1. Definition. A function $u \in V$ satisfying the identity

$$\sum_{|\alpha| \leqslant m} \int_\Omega [A_\alpha(x, u, ..., D^m u) - f_\alpha(x)] \, D^\alpha \varphi(x) \, dx = 0$$

for any $\varphi \in V$ is called the *weak solution* of the boundary value problem for Eq. (1.1) on the space V.

5.2. *Remark.* Let us note that any weak solution of (1.1) on V satisfies the operator equation

$$Tu = 0, \tag{5.1}$$

where $T: V \to V^*$ is the operator defined in (3.1).

5.3. Theorem. *Let us suppose that* (a)–(d) *from Section* 3.1, (i)–(iv) *from Section* 3.2, (4.1), *and* (4.2) *hold. Moreover, assume that there exists a number* $R > 0$ *such that*

$$\sum_{|\alpha| \leqslant m} \int_\Omega [A_\alpha(x, u, ..., D^m u) - f_\alpha(x)] \, D^\alpha u(x) \, dx \geqslant 0 \tag{5.2}$$

holds for all $u \in V$ *satisfying* $\|u\|_{m,p,\nu} = R$. *Then the boundary value problem for Eq.* (1.1) *on* V *has at least one weak solution* $u_0 \in V$ *such that* $\|u_0\|_{m,p,\nu} \leqslant R$.

Proof. Condition (5.2) is equivalent to

$$\langle Tu, u \rangle \geqslant 0. \tag{5.3}$$

If there is $\|u_0\|_{m,p,\nu} = R$ such that $Tu_0 = 0$ the assertion is proved. In the opposite case it follows from (5.3) and Theorem 1.3.4 in [14] that

$$\operatorname{Deg}(T; B_R(0), 0) = 1,$$

where $B_R(0)$ is the open ball in V centered at the origin and with radius $R > 0$. The basic property of the degree [14, Corollary 1.3.1] implies the existence of at least one $u_0 \in B_R(0)$ such that $Tu_0 = 0$.

5.4. *The Asymptotically Homogeneous Case*

Assume that $A_\alpha^{(i)}(x, \xi)$, $i = 0, 1$, are Carathéodory functions for any $|\alpha| \leqslant m$. Let

$$A_\alpha(x, \xi) = A_\alpha^{(0)}(x, \xi) + A_\alpha^{(1)}(x, \xi)$$

and assume that the following conditions are satisfied.

(I) $A_\alpha^{(0)}$ satisfy the growth assumptions (i)–(iv) from Section 3.2.

(II) $A_\alpha^{(1)}$ satisfy the growth assumptions

$$|A_\alpha^{(1)}(x,\xi)| \leqslant c v_\alpha^{1/p}\left[1+\sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} |\xi_\beta|^{\tilde p} + \sum_{|\beta|=m} v_\beta^{\tilde p/p} |\xi_\beta|^{\tilde p}\right]$$

for $|\alpha| = m$;

$$|A_\alpha^{(1)}(x,\xi)| \leqslant c\left[1+\sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} |\xi_\beta|^{\tilde p} + \sum_{|\beta|=m} v_\beta^{\tilde p/p} |\xi_\beta|^{\tilde p}\right]$$

for $|\alpha| \leqslant m-1$, with some $\tilde p < p-1$ and a constant $c > 0$.

(III) $A_\alpha^{(0)}(x,\xi)$ are *positively homogeneous* in ξ of order $p-1$ and odd in ξ, i.e.,

$$A_\alpha^{(0)}(x, t\xi) = t^{p-1} A_\alpha^{(0)}(x,\xi) \qquad \text{for any} \quad t>0,\ \xi \in \mathbb{R}^{M(m)};$$
$$A_\alpha^{(0)}(x, -\xi) = -A_\alpha^{(0)}(x,\xi) \qquad \text{for any} \quad \xi \in \mathbb{R}^{M(m)}.$$

(IV) For a.e. $x \in \Omega$ and for all $\zeta = \{\zeta_\alpha; |\alpha| = m\} \in \mathbb{R}^{M_0}$, $\zeta' = \{\zeta'_\alpha; |\alpha| = m\} \in \mathbb{R}^{M_0}$, $\eta = \{\eta_\alpha; |\alpha| < m\} \in \mathbb{R}^{M(m-1)}$, $\zeta \neq \zeta'$ the following two inequalities hold:

$$\sum_{|\alpha|=m} [A_\alpha(x,\eta,\zeta) - A_\alpha(x,\eta,\zeta')](\zeta_\alpha - \zeta'_\alpha) > 0,$$
$$\sum_{|\alpha|=m} [A_\alpha^{(0)}(x,\eta,\zeta) - A_\alpha^{(0)}(x,\eta,\zeta')](\zeta_\alpha - \zeta'_\alpha) > 0.$$

(V) $A_\alpha^{(0)}$ satisfy the ellipticity condition (4.1).

5.5. THEOREM. *Assume that the above conditions* (I)–(V) *are fulfilled and that the boundary value problem for*

$$\sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha^{(0)}(x, u, ..., D^m u) = 0 \tag{5.4}$$

on space V has only a trivial solution. Then the boundary value problem for Eq. (1.1) *on space V has at least one weak solution for arbitrary functions f_α (on the right-hand side of* (1.1)) *satisfying* (a)–(d) *from Section* 3.1.

Proof. It follows from (I)–(V) that the operators A, $A_0\colon V \to V^*$ defined by

$$\langle Au, \varphi\rangle = \sum_{|\alpha| \leqslant m} \int_\Omega A_\alpha(x, u, ..., D^m u)\, D^\alpha\varphi(x)\, dx,$$

$$\langle A_0 u, \varphi\rangle = \sum_{|\alpha| \leqslant m} \int_\Omega A_\alpha^{(0)}(x, u, ..., D^m u)\, D^\alpha\varphi(x)\, dx$$

are bounded and continuous and satisfy condition $\alpha(V)$. The conditions (I), (II), (III), and (5.4) imply that A is an asymptotically homogeneous operator regular at infinity (see [14]). It follows from (III) that A_0 is an odd operator and hence due to (5.4) the index of zero of the mapping A_0 is different from zero. Then it follows from Theorem 1.6.3 in [14] that the equation $Au=h$ is solvable for any $h\in V^*$. But this operator equation is equivalent to the boundary value problem for Eq. (1.1) on the space V.

5.6. THEOREM. *Let us suppose that the assumptions* (i)–(iv) *from Section* 3.2, (4.1), *and* (4.2) *hold. Moreover, let us suppose that the following coercivity condition is fulfilled*:

$$\sum_{|\alpha|\leqslant m} A_\alpha(x,\xi)\,\xi_\alpha \geqslant c_1 \sum_{|\alpha|=m} v_\alpha(x)\,|\xi_\alpha|^p + c_2\,|\xi_{\alpha_0}|^p - c_3 \tag{5.5}$$

for a.e. $x\in\Omega$ *and for all* $\xi\in\mathbb{R}^{M(m)}$ *where* $\alpha_0=(0,\dots,0)$ *and* c_1, c_2, c_3 *are positive constants. Then the boundary value problem for Eq.* (1.1) *on space* V *has at least one weak solution for arbitrary functions* f_α *satisfying* (a)–(d) *from Section* 3.1.

Proof. It follows from the compact imbedding of $W^{m,p}(v,\Omega)$ into $W^{m-1,p}(\Omega)$ and from [11, Proposition 4.1, p. 59] that for any $\varepsilon>0$ there exists $c_\varepsilon>0$ such that

$$\begin{aligned}\Bigg(\sum_{|\beta|\leqslant m-1}\int_\Omega |D^\beta u|^p\,dx\Bigg)^{1/p} &\leqslant \varepsilon\Bigg[\sum_{|\beta|\leqslant m-1}\int_\Omega |D^\beta u|^p\,dx + \sum_{|\beta|=m}\int_\Omega v_\beta(x)\,|D^\beta u|^p\,dx\Bigg]^{1/p}\\ &\quad + c_\varepsilon\Bigg(\int_\Omega |u|^p\,dx\Bigg)^{1/p}.\end{aligned}$$

Elementary calculation yields that there is a constant $c>0$ such that

$$\sum_{|\beta|\leqslant m-1}\int_\Omega |D^\beta u|^p\,dx \leqslant c\Bigg(\sum_{|\beta|=m}\int_\Omega v_\beta(x)\,|D^\beta u|^p\,dx + \int_\Omega |u|^p\,dx\Bigg).$$

In particular, this means that there is a constant $\tilde c>0$ such that

$$\|u\|^p_{m,p,v} \leqslant \tilde c\Bigg(\sum_{|\beta|=m}\int_\Omega v_\beta(x)\,|D^\beta u|^p\,dx + \int_\Omega |u|^p\,dx\Bigg). \tag{5.6}$$

It means that the term on the right-hand side of (5.6) defines an equivalent norm on V.

The assumptions (5.5) together with (5.6) yield

$$\begin{aligned}
&\int_\Omega \sum_{|\alpha|\leqslant m} A_\alpha(x, u, ..., D^m u)\, D^\alpha u\, dx \\
&\quad \geqslant c_1 \sum_{|\alpha|=m} \int_\Omega v_\alpha(x)\, |D^\alpha u(x)|^p\, dx + c_2 \int_\Omega |u(x)|^p\, dx - c_3 \\
&\quad \geqslant \min\{c_1, c_2\} \left[\sum_{|\alpha|=m} \int_\Omega v_\alpha(x)\, |D^\alpha u(x)|^p\, dx \right. \\
&\quad\quad \left. + \int_\Omega |u(x)|^p\, dx \right] - c_3 \geqslant \text{const } \|u\|^p_{m,p,v} - c_3.
\end{aligned} \tag{5.7}$$

Let f_α be arbitrary functions satisfying (a)–(d) from Section 3.1. Then due to (5.7) the operator T defined in (3.1) is *coercive* on V, i.e.,

$$\frac{\langle Tu, u\rangle}{\|u\|_{m,p,v}} \to \infty \qquad \text{for} \quad \|u\|_{m,p,v} \to \infty.$$

Hence the hypotheses of Theorem 5.3 are verified and the proof is completed.

5.7. Example. Let us consider for Ω the plane domain (i.e., $n = 2$) $\Omega =]-1, 1[\times]-1, 1[$ and put for $x = (x_1, x_2) \in \Omega$

$$v(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 \leqslant 0, \\ x_2^\lambda (1 - x_1)^\gamma & \text{for } x_1 > 0, x_2 > 0, \\ |x_2|^\mu (1 - x_1)^\gamma & \text{for } x_1 > 0, x_2 < 0 \end{cases} \tag{5.8}$$

with λ, μ, γ real numbers.

(i) We consider the differential operator

$$(Au)(x) = (-1)^m \sum_{|\alpha|=m} D^\alpha(v(x)\, |D^\alpha u(x)|^{p-2}\, D^\alpha u(x)) \tag{5.9}$$

with $v(x) = v(x_1, x_2)$ from (5.8) and with $1 < p < \infty$. Thus we have a (generally nonlinear) operator of order $2m$ with coefficients $v(x)$ whose *degeneration* (or *singularity*) is concentrated on a part Γ_1 of the boundary $\partial\Omega$,

$$\Gamma_1 = \{x = (x_1, x_2);\, x_1 = 1,\, x_2 \in]-1, 1[\},$$

as well as on a segment Γ_2 in the interior of Ω,

$$\Gamma_2 = \{x = (x_1, x_2);\, x_1 \in]0, 1[,\, x_2 = 0\}.$$

(The reader is invited to sketch a figure).

Conditions (2.6) and (2.7) indicate that we have to choose λ, μ from the interval $]-1, p-1[$ with no condition on γ. Condition (2.11) indicates that λ, μ, and γ must be less than $p/2$, so that finally (taking into account (2.12)) we must assume

$$\begin{aligned} &-1<\lambda,\ \mu<\min\left\{\frac{p}{2}, p-1\right\}=\begin{cases} p-1 & \text{for} \quad 1<p\leqslant 2, \\ \dfrac{p}{2} & \text{for} \quad p\geqslant 2, \end{cases} \\ &\gamma<\min\left\{\frac{p}{2}, p-1\right\}. \end{aligned} \tag{5.10}$$

For λ, μ, and γ *positive* we have a degeneration which will be very small for p close to 1. On the other hand, also a singularity can occur, in a limited extend at Γ_2 (for λ or μ *negative*, but bigger than -1), but big enough at Γ_1 (for $\gamma \in]-\infty, 0[$).

In our case, it is

$$\begin{aligned} A_\alpha(x, \xi) &= v(x)\, |\xi_\alpha|^{p-2}\, \xi_\alpha \qquad \text{for} \quad |\alpha| = m, \\ A_\alpha(x, \xi) &= 0 \qquad\qquad\qquad\quad \text{for} \quad |\alpha| < m, \end{aligned} \tag{5.11}$$

and obviously, the growth conditions from Section 3.2 [more precisely, the condition 3.2(i)] and the ellipticity and monotonicity conditions (4.1) and (4.2) are satisfied (with the functions $g_i(t) = 1$, $i = 1, 2, 3$). The appropriate space $W^{m,p}(v, \Omega)$ is the set

$$\left\{ u = u(x);\ \int_\Omega |D^\alpha u(x)|^p\, v(x)\, dx < \infty \quad \text{for} \quad |\alpha| = m, \right.$$
$$\left. \int_\Omega |D^\beta u(x)|^p\, dx < \infty \quad \text{for} \quad |\beta| < m \right\}.$$

Let us now check the number κ_1 from (2.10). According to (2.12) $g^* > \max\{2/p, 1/(p-1)\}$. If we choose $1 < p < 2$, then we can take

$$g^* = \frac{1}{p-1} + \varepsilon, \qquad \varepsilon > 0,$$

and obtain that

$$\kappa_1 = m - \frac{2}{p_1} = m - \frac{2}{p}\,\frac{p + \varepsilon(p-1)}{1 + \varepsilon(p-1)},$$

i.e.,

$$\kappa_1 \in \left]m-2, m-\frac{2}{p}\right[\qquad \text{for} \quad 1<p<2. \tag{5.12}$$

If we choose $p>2$, then we can take $g^* = 2/p + \varepsilon$, $\varepsilon > 0$, and obtain that $\kappa_1 = m - 2/p_1 = m - 2/p - 2/(2+\varepsilon p)$, i.e.,

$$\kappa_1 \in \left]m-1-\frac{2}{p}, m-\frac{2}{p}\right[\qquad \text{for} \quad p>2. \tag{5.13}$$

(ii) Let us suppose that the number κ_1 is positive and consider the differential operator

$$(Au)(x) + \sum_{|\alpha|<\kappa_1} (-1)^{|\alpha|} D^\alpha(g_\alpha(|D^\alpha u|)\, sgn\, D^\alpha u),$$

where A is the operator from (5.9) and $g_\alpha : \mathbb{R} \to \mathbb{R}^+$ are continuous and nondecreasing functions. In addition to part (i) of this example, we only have to check whether the growth condition 3.2 (iv) is satisfied, but this is obviously the case, since here,

$$A_\alpha(x, \xi) = g_\alpha(|\xi_\alpha|)\, sgn\, \xi_\alpha, \qquad |\alpha| < \kappa_1. \tag{5.14}$$

For example, if $m=1$ and $p>2$ in the principal part, we can take $g_\alpha(t) = te^{t^2}$ for $|\alpha| < \kappa_1$ and consider the equation

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(v(x_1, x_2)\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + ue^{u^2} = f \qquad \text{in } \Omega.$$

Here, the assumption $p>2$ is essential, since for $m=1$ and $1<p<2$, the admissible values of κ_1 are negative (cf. (5.12)) and the set $\xi_0 = \{\xi_\beta ; |\beta| < \kappa_1\}$ is empty.

The functions A_α defined by (5.11) for $\kappa_1 \leqslant |\alpha| \leqslant m$ and by (5.14) for $|\alpha| < \kappa_1$ satisfy coercivity assumption (5.5) and hence we obtain the existence result applying Theorem 5.6.

5.8. *Remark.* Let us remark that the growth assumptions (i)–(iv) from Section 3.2 concerning lower-order terms are more general than those in [5, 3]. On the other hand also the case of unbounded domain Ω is studied in [3, 5] and the boundedness in $L^\infty(\Omega)$ of the weak solution is proved in these papers. The price we must pay for the study of higher-order equations is the fact that the truncation method due to De Giorgi (which was applied in [3, 5]) is not possible to apply here.

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